On Optimal Solution of the Compromise Ranking Problem

Jean Dezert
The French Aerospace Lab
Palaiseau, France.
jean.dezert@onera.fr

Andrii Shekhovtsov, Wojciech Sałabun
National Telecom. Institute
Warsaw, Poland.
a.shekhovtsov@il-pib.pl, w.salabun@il-pib.pl

Albena Tchamova
Inst. of I&C Tech., Bulg. Acad. of Sciences
Sofia, Bulgaria.
tchamova@bas.bg

***Abstract*—This paper is about the Compromise Ranking Problem (CRP), a well-known problem in the social choice theory. According to the famous Arrow's theorem there is no voting method which is entirely satisfying and fairness if one accepts Arrow's axioms. In this paper we formalize the problem as a minimisation problem in a discrete finite search space. We attempt to solve it based on the Least Squares (LS) approach thanks to some appealing metrics to get the optimal CRP solution. Surprisingly, we show that the optimal consensus (compromise) ranking solution disagrees with the commonsense solutions in four simple interesting examples. The search for an optimal solution in agreement with the commonsense appears to be an open very challenging question and our paper warns the users about the impossibility of the main current methods to provide acceptable solutions even for the rather simple examples considered in this work.**

Keywords: preferences fusion, compromise ranking, total preference orderings, Kemeny distance, Frobenius distance, decision-making.

I. Introduction

This paper is about Compromise Ranking Problem (CRP) also called the ranking aggregation problem, or the ranking fusion problem in the literature. This is a well-known important problem of the social choice theory (i.e. the science of elections) [1], [2], which studies how a society should choose among a set of various alternatives[1] (or options) based on the preferences of its individual members. Borda [3] and Condorcet [4] in the late eighteenth century proposed methodologies to make the social choice which are unfortunately not exempt of shortcomings. The first most influential work has been achieved in the mid of 20th century by Arrow[2] in his milestone book [5] on *Social Choice and Individual Values* where he defines a *Social Welfare Function* (SWF) as any rule for determining the society's preferences over the (social) alternatives from the knowledge of the preferences of the individual members of the society. The fascinating Arrow's result is his famous Impossibility Theorem (IT) which establishes that if a society has at least two members and three options to choose from, then no SWF can jointly meet the following four reasonable expected desiderata (i.e. Arrow's axioms):

1) **D1**: (*Unrestricted Domain*) The SWF must be able to accept as inputs all possible preference orderings expressed by the members of the society.
2) **D2**: (*Unanimity*[3]) If all members prefer option A to option B, then the SWF must rank A over B[4].
3) **D3**: (*Non-Dictatorship*) The SWF cannot have as its output the preference orderings of a member, for all possible preference orderings of that member, i.e. a member must not dictate his own preference ordering.
4) **D4**: (*Independence of irrelevant alternatives*[5]) Including a new alternative in the existing set of alternatives must not impact the preference ordering of the existing alternatives.

Arrow showed that if a SWF satisfies D1, D2 and D4 desiderata then it must be a dictatorship (i.e. we must have as its output only the preference orderings of a single individual who is the dictator). This important Arrow's theorem is frustrating for setting solid bases of democracies and for social decision-making in general if we consider these four desiderata as bases for a democracy. The desiderata D1, D2 and D3 are considered as very reasonable and necessary, but the desideratum D4 is controversial and not unanimously considered as a necessary criterion. The violation of an Arrow's desideratum diminishes the desirability of the SWF function and many researchers working in social choice theory consider that D4 is the most acceptable desideratum to violate because D4 might appear an unrealistically constraining, since it prohibits information about the intensity (or weights) of individuals' preferences for the available choice[6]. If we take into account preference intensity information in a SWF, and we can make meaningful comparisons of such information, then we can avoid Arrow's impossibility theorem [6]. However there are serious difficulties with trying to take account of such intensities when conducting elections[7]. In this work, the CRP

[1]The *alternatives* terminology is a generic here. They refer to candidates in an election, physical objects of a list, or set of hypotheses, etc., depending on the problem under concern.

[2]1972 Nobel Memorial Prize in Economic Sciences.

[3]Also named the Pareto property.

[4]denoted as $A \succ B$ to mean that A is preferred to B.

[5]Also known as Rank Reversal paradox.

[6]For instance in a two-member society, if individual 1 prefers A to B and individual 2 prefers B to A in each of two preference orderings, then D4 requires that the social preferences between A and B be the same for both preference orderings (denoted by $A \equiv B$) despite the possibility that in one preference ordering, individual 1 may have a strong preference for A over B and individual 2 may have only a slight preference for B over A.

[7]as mentioned in Masking's Preface of the 2012 3rd edition of [5].

statement focuses on purely rank-order information without any finer grained notion of weighted preferences because the information about the intensity (or weights) of individuals' preferences for the choice is rarely available, or very difficult to obtain precisely. In contrast to Arrow's axiomatic approach of the social choice theory (including weakenings of its axioms to avoid the impossibility theorem), we propose a non-axiomatic approach for making the social choice in order to get fusion of rankings. Our proposed method is mainly based on Kemeny's and Frobenius distances between preference orderings, and we express the CRP as a least squares optimization problem in a discrete search space [7]. This approach should be able (at least in theory) to deal with any type of Total Preference Orderings (TPOs) including those having ties, and we tolerate that the optimal social choice solution can also include ties. We do not discuss in this paper the tie-breaking strategies[8] in case of tied-top ranked alternatives which is out of the scope of this paper. In this work we address CRP when dealing with TPOs with possibility of ties, which means that all alternatives in competition enter in the expression of the Preference Orderings (PO) of each member of the society[9]. In this paper we clearly show that the optimal LS solution of the CRP when dealing with TPOs is far from satisfactory, and that more research efforts have to be carried out. This negative result is not vain because it clearly demonstrates that classical methods for rank aggregation can easily fail to provide reasonable and acceptable solutions, as well as the Kemeny's Optimal Approach (KOA) and Frobenius' Optimal Approach (FOA) in their current stage of development.

This paper is organized as follows. In section II we briefly introduce some notations and present the CRP that we address in this paper. Section III defines the discrete space of the solutions of CRP, its dimension with some examples. Section IV briefly presents classical methods to generate (non-optimal) solution of CRP. Section V formalizes the CRP as a general discrete optimization problem based on metrics, with examples. We show the impossibility of KOA and FOA to provide the expected solutions for all simple examples we consider in this paper. Section VI concludes the paper with perspectives and challenging open questions for future research directions.

II. THE COMPROMISE RANKING PROBLEM (CRP)

The CRP (i.e. the ranking aggregation problem) concerns the combination of several rankings (or preference orderings[10]) in order to obtain a final ranking that satisfies specific criteria. Let's consider a set $X = \{x_1, x_2, \dots, x_n\}$ of n distinct *alternatives* (also named *competitors*, or *options* in the literature). Let's suppose that we have S distinct sources of information (i.e. evidences) E_1, E_2, …, and E_S (i.e. the members of a "Society") that provide their total preference orderings Pref_1, Pref_2, …, Pref_S about these alternatives with eventually ties. The CRP we address in this paper is on how to obtain the optimal (in some sense) global aggregated TPO $\text{Pref}^\star$ result that would summarize the whole set of preferences for making a final decision. In other words how to build a preference fusion/aggregation rule or method denoted symbolically $F(\text{Pref}_1, \dots, \text{Pref}_S)$ such that $\text{Pref}^\star = F(\text{Pref}_1, \dots, \text{Pref}_S)$ in order to make the final decision about the best alternative (or best tied alternatives if any) located at the first rank to select?

For instance, consider the following examples:

- **Example 1**: Three alternatives A, B, and C, and two members with the following TPOs

$$\text{Pref}_1: \ A \succ C \succ B, \quad \text{Pref}_2: \ B \succ C \succ A.$$

This example is inspired from the well-known Zadeh's example [8] used to question the validity of Dempster's rule of combination in Dempster-Shafer Theory [9].

- **Example 2**: Three alternatives A, B, and C, and three members with the following TPOs

$$\text{Pref}_1 : A \succ B \succ C, \quad \text{Pref}_2 : B \succ C \succ A,$$
$$\text{Pref}_3 : C \succ A \succ B.$$

This is the famous Condorcet's paradox example [4], [5], where no strict social ordering and clear rational decision can be drawn to obtain a top winner at the first rank.

- **Example 3**: Three alternatives A, B, and C, and four members with the following TPOs

$$\text{Pref}_1: \ A \succ B \succ C, \quad \text{Pref}_2: \ A \succ B \succ C,$$
$$\text{Pref}_3: \ A \succ B \succ C, \quad \text{Pref}_4: \ C \succ A \succ B.$$

- **Example 4**: Four alternatives A, B, C, and D, and three members with the following TPOs

$$\text{Pref}_1 : A \succ B \succ C \succ D, \quad \text{Pref}_2 : D \succ (B \equiv A) \succ C,$$
$$\text{Pref}_3 : B \succ C \succ A \succ D.$$

What should be the expected social choice (compromise ranking) $\text{Pref}^\star$ for these rather simple examples?

The expected (rational) solution for example 1 is $\text{Pref}^\star : (A \equiv B) \succ C$ because from $\text{Pref}_1 : A \succ C \succ B$ the alternative A is the winner (the one at the top 1st rank), and from $\text{Pref}_2 : \ B \succ C \succ A$ the alternative B is the winner. Hence, this means that we must consider A and B as ex-aequo winners, represented by the tie $(A \equiv B)$. In this example there is no way to break the ambiguous symmetries based on the only information we have for breaking the tie. It is not rational to conclude that $C \succ (A \equiv B)$ is valid[11] because both experts agree that C must not be the winner. So C must be eliminated

[8] using instant runoff value, minimax and the Borda count for instance.

[9] When some alternatives are missing in a Preference Ordering we say that we have a Partial Preference Ordering (PPO). For instance if we have three possible alternatives A, B and C, then $A \succ B \succ C$ and $(A \equiv B) \succ C$ are two possible TPOs, whereas $A \succ B$ and $(A \equiv B)$ are only PPOs.

[10] We use indifferently the terminology *ranking* and *preference orderings* because they are one-to-one. For instance, given an (ordered) set of alternatives $\{A, B, C\}$ the ranking vector $\mathbf{r} = [r(A), r(B), r(C)] = [3, 1, 2]$ is equivalent to the preference ordering $B \succ C \succ A$, and vice-versa.

[11] This is what discussed Zadeh in his famous example [8] against the validity of Dempster's rule which will consider $C \succ (A \equiv B)$ valid. Another very serious and more general counter-example against Dempster's rule can be found in [10], [11].

and what logically remains is the ex-aequo winners A and B in such situation.

The expected natural (rational) solution for example 2 needs more attention because in this Condorcet's paradox example no single alternative can be the winner w.r.t the others, neither two tied alternatives can be the definitive winner because of the circularity of alternatives in the three TPOs. In this Condorcet's example the only reasonable (or rational) solution is to take $\text{Pref}^\star : (A \equiv B \equiv C)$ which characterizes the full uncertain situation for decision-making. Actually the three alternatives are tied altogether at the 1st rank because there is not enough evidence in TPOs to break this tie, unless some ad-hoc (and always disputable) heuristic method is used. The expected natural (rational) solution for example 3 should be $\text{Pref}^\star : A \succ B \succ C$ because A is the top winner at rank 1, then after eliminating A of Pref_1, Pref_2, Pref_3 and Pref_4 and examining them makes B as the top winner for rank 2, and put C at rank 3. The expected natural (rational) solution for example 4 is less obvious/immediate and we expect to get $(A \equiv B \equiv D) \succ C$. The solution of these examples based on KOA and FOA will be discussed in Sections V. At this stage we already anticipate the difficulty of finding the best compromise ranking solution in more general (bigger) problems involving many alternatives and many members of the society under concern.

III. The search space of the social choice solution

Consider $n > 1$ distinct alternatives (i.e. elements in competition, candidates in an election, objects, etc). We denote by $\mathcal{P}(n)$ the set of all possible Total Preference Orderings (TPOs) that corresponds to the number of ways n objects can rank in a competition, allowing for the possibility of ties. The dimension of the search space[12] $\mathcal{P}(n)$ for a given number of alternatives $n > 1$ where CR solution must belong grows much faster than 2^n as it will be shown in the next tables and by the formula of $|\mathcal{P}(n)|$ in (2).

To evaluate the cardinality of $\mathcal{P}(n)$, let us make a quick analysis on how many preferences orderings (allowing for the possibility of ties) we have for two, three and four alternatives respectively. For two alternatives A and B the set $\mathcal{P}(2)$ has obviously only three TPOs as listed in Table I. For three alternatives A, B and C the set $\mathcal{P}(3)$ includes 13 TPOs as listed in Table II.

Table I: Set $\mathcal{P}(2)$ for 2 alternatives.

$A \succ B$
$B \succ A$
$(A \equiv B)$

Table II: Set $\mathcal{P}(3)$ for 3 alternatives.

$A \succ B \succ C$	$A \succ (B \equiv C)$
$A \succ C \succ B$	$B \succ (A \equiv C)$
$B \succ A \succ C$	$C \succ (A \equiv B)$
$B \succ C \succ A$	$(A \equiv B) \succ C)$
$C \succ A \succ B$	$(A \equiv C) \succ B)$
$C \succ B \succ A$	$(B \equiv C) \succ A)$
	$(A \equiv B \equiv C)$

For four alternatives A, B, C and D the set $\mathcal{P}(4)$ is more difficult to establish manually because it includes 75 TPOs. This set $\mathcal{P}(4)$ can decomposed into a subset of 24 strict TPOs (see Table III), a subset of 8 tied TPOs with three tied objects among four (see Table IV), a subset of 6 tied TPOs with four objects tied two-by-two (see Table V), a subset of 36 tied TPOs with two tied objects among four (see Table VI), and a subset of only one TPO with the tie of all four elements (i.e. $(A \equiv B \equiv C \equiv D)$). Hence the cardinality of $\mathcal{P}(4)$ is $|\mathcal{P}(4)| = 24 + 8 + 6 + 36 + 1 = 75$.

Table III: 24 Strict (non-tied) TPOs for 4 alternatives.

$A \succ B \succ C \succ D$	$B \succ A \succ C \succ D$
$A \succ B \succ D \succ C$	$B \succ A \succ D \succ C$
$A \succ C \succ D \succ B$	$B \succ C \succ D \succ A$
$A \succ C \succ B \succ D$	$B \succ C \succ A \succ D$
$A \succ D \succ B \succ C$	$B \succ D \succ A \succ C$
$A \succ D \succ C \succ B$	$B \succ D \succ C \succ A$
$C \succ A \succ B \succ D$	$D \succ A \succ B \succ C$
$C \succ A \succ D \succ B$	$D \succ A \succ C \succ B$
$C \succ B \succ D \succ A$	$D \succ B \succ C \succ A$
$C \succ B \succ A \succ D$	$D \succ B \succ A \succ C$
$C \succ D \succ A \succ B$	$D \succ C \succ A \succ B$
$C \succ D \succ B \succ A$	$D \succ C \succ B \succ A$

Table IV: 8 Tied Total Preference Orderings (TTPOs) with three tied alternatives among four.

$A \succ (B \equiv C \equiv D)$	$(B \equiv C \equiv D) \succ A$
$B \succ (A \equiv C \equiv D)$	$(A \equiv C \equiv D) \succ B$
$C \succ (A \equiv B \equiv D)$	$(A \equiv B \equiv D) \succ C$
$D \succ (A \equiv B \equiv C)$	$(A \equiv B \equiv C) \succ D$

Table V: 6 Tied Total Preference Orderings (TTPOs) with four alternatives tied two-by-two.

$(A \equiv B) \succ (C \equiv D)$	$(C \equiv D) \succ (A \equiv B)$
$(A \equiv C) \succ (B \equiv D)$	$(B \equiv D) \succ (A \equiv C)$
$(A \equiv D) \succ (B \equiv C)$	$(B \equiv C) \succ (A \equiv D)$

Table VI: 36 Tied Total Preference Orderings (TTPOs) with two tied alternatives among four.

$(A \equiv B) \succ C \succ D$	$C \succ (A \equiv B) \succ D)$	$C \succ D \succ (A \equiv B)$
$(A \equiv B) \succ D \succ C$	$D \succ (A \equiv B) \succ C)$	$D \succ C \succ (A \equiv B)$
$(A \equiv C) \succ B \succ D$	$B \succ (A \equiv C) \succ D)$	$B \succ D \succ (A \equiv C)$
$(A \equiv C) \succ D \succ B$	$D \succ (A \equiv C) \succ B)$	$D \succ B \succ (A \equiv C)$
$(A \equiv D) \succ B \succ C$	$B \succ (A \equiv D) \succ C)$	$B \succ C \succ (A \equiv D)$
$(A \equiv D) \succ C \succ B$	$C \succ (A \equiv D) \succ B)$	$C \succ B \succ (A \equiv D)$
$(B \equiv C) \succ A \succ D$	$A \succ (B \equiv C) \succ D)$	$A \succ D \succ (B \equiv C)$
$(B \equiv C) \succ D \succ A$	$D \succ (B \equiv C) \succ A)$	$D \succ A \succ (B \equiv C)$
$(B \equiv D) \succ A \succ C$	$A \succ (B \equiv D) \succ C)$	$A \succ C \succ (B \equiv D)$
$(B \equiv D) \succ C \succ A$	$C \succ (B \equiv D) \succ A)$	$C \succ A \succ (B \equiv D)$
$(C \equiv D) \succ A \succ B$	$A \succ (C \equiv D) \succ B)$	$A \succ B \succ (C \equiv D)$
$(C \equiv D) \succ B \succ A$	$B \succ (C \equiv D) \succ A)$	$B \succ A \succ (C \equiv D)$

Actually the number $|\mathcal{P}(n)|$ of preferential arrangements of n labeled elements (i.e. alternatives) is the number of distinct weak orders (represented either as strict weak orders or as total preorders) on an n-element set. $|\mathcal{P}(n)|$ follows the ordered Bell numbers[13] sequence 3, 13, 75, 541, 4683, 47293,

[12]It corresponds to the set of all the ordered partitions of a set, which is also named the *ordered set partitions*.

[13]also named Fubini's numbers in the literature [12].

545835, …(see [13], [14], and the sequence 1191 in [15] which corresponds to the sequence A000670 in the On-line Encyclopedia of Integer Sequences (OEIS) [16] with a recent discussion on OEIS in [17]). Because of the high dimension of the search space $\mathcal{P}(n)$ for $n > 4$ we will present some examples only for $n \leq 4$ in this paper. In 1962 Gross[14] derived in [18] (p. 7) the following recurrent formula of $|\mathcal{P}(n)|$

$$|\mathcal{P}(n)| = 1 + \sum_{k=1}^{n-1} \binom{n}{k} |\mathcal{P}(n-k)| \tag{1}$$

where $\binom{n}{k} = \frac{n!}{(n-k)!k!}$ is the number of k-combinations of an n-set (i.e. a binomial coefficient[15]), and $|\mathcal{P}(0)| = 1$. As explained by Pippenger in [19] (p. 338), for $n \geq 1$, we can construct a preferential arrangement on n alternatives by first choosing the number k of alternatives tied in the top equivalence class (with k in the range $1 \leq k \leq n$), then choosing in one of n ways k alternatives in this class, and finally choosing in one of $|\mathcal{P}(n-k)|$ ways a preferential arrangement of the remaining $n-k$ candidates. Another nice explicit and direct formula derived by Mendelson [20] based on inclusion-exclusion principle is

$$|\mathcal{P}(n)| = \sum_{k=1}^{n} \sum_{i=1}^{k} (-1)^{k-i} \binom{k}{i} i^n \tag{2}$$

Interesting additional discussions on the derivation of $|\mathcal{P}(n)|$ from generating function $(2-e^x)^{-1}$ can be found in [21]. Some algorithms for generating the set $\mathcal{P}(n)$ of preferential arrangements are presented in [22], and all ordered partitions can be generated in Matlab™ from the list of non ordered partitions obtained with Luong's code [23] for instance. For Python users, codes for generating set partitions are provided on GitHub in the Sage Mathematical Software Systems [24].

IV. Classical methods

Because preference ordering and ranking are one-to-one, the problem of preferences aggregation is equivalent to the problem of rankings aggregation and many methods have been devoted to this problem specially in the social choice theory, multi-criteria decision-making support, meta-search engines, etc. We use indifferently the preference formalism or the ranking formalism depending on the method we present. We briefly recall here the three main classical methods[16] encountered in many applications.

A. Borda's counting method

Consider n alternatives $\{x_1, \ldots, x_n\}$ to rank from a set of S rankings or preferences Pref_i ($i = 1, \ldots, S$). Then, for each Pref_i an alternative receives n points (denoted $pt(.)$) if it appears at the 1st rank, $n-1$ points if it appears at the 2nd rank, and so on. The points are then summed and the alternative with the most points wins. Borda's method [3] does not satisfy Arrow's desideratum D4, and it does not always provide the expected/result when applied. For instance, in our example 1 we get $pt_1(A) = 3$, $pt_1(C) = 2$ and $pt_1(B) = 1$ from $\text{Pref}_1 : \ A \succ C \succ B$, and we get $pt_2(B) = 3$, $pt_2(C) = 2$ and $pt_2(A) = 1$ from $\text{Pref}_2 : \ B \succ C \succ A$. Hence the Borda's score for each alternative is $pt_1(A) + pt_2(A) = 4$, $pt_1(B) + pt_2(B) = 4$ and $pt_1(C) + pt_2(C) = 4$, which means that alternatives A, B and C are ex-aequo according to Borda. Hence Borda's solution would be the 3-alternatives tie ($A \equiv B \equiv C$), which is clearly counter-intuitive. Borda's method is equivalent to add the rank values of a given alternative in all the preference orderings to get a score, and order the scores in ascending order (the smallest value being the most preferred alternative).

B. Copeland's method

This method[17] [26] is a ranked-voting method based on scoring pairwise wins and losses. It belongs to the class of Condorcet methods[18] [28] which are based on the pairwise comparisons of alternatives (i.e. 1-to-1 match-ups) in each preference ordering. Copeland's method may give rise to tied results. Each voter (the source of evidence) must give an ordered preference list on candidates where ties are allowed. A result $n \times n$ matrix $\boldsymbol{M} = [m_{ij}]$ is constructed with $m_{ij} = 1$ if more voters strictly prefers alternative x_i to x_j, $m_{ij} = 0$ if more voters strictly prefers alternative x_j to x_i, and $m_{ij} = 0.5$ otherwise. By convention m_{ii} equals 0. The score for candidate x_i is the sum over j of the m_{ij}. If a candidate gets a score of $n-1$ then this candidate is the (necessarily unique) winner. Otherwise the method produces no clear decision and the candidates with greatest score value are the Copeland tied winners. This method satisfies Condorcet's criterion, i.e. if a candidate would win against each of their rivals in a one-on-one vote, this candidate is the winner, but it does not mean that the result is always rational (i.e. it makes sense) in other situations as shown for our example 1. Because for our example 1 we obtain the $\boldsymbol{M}$ matrix and score vector $\boldsymbol{s}$ as follows

$$\boldsymbol{M} = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{bmatrix}, \text{ and } \boldsymbol{s} = \begin{bmatrix} s(A) \\ s(B) \\ s(C) \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{3}$$

Based on $\boldsymbol{s}$, we see that all score values are equal which means that alternatives A, B and C are ex-aequo according to Copeland's method. Hence this solution would be the 3-alternatives tie ($A \equiv B \equiv C$), which is also clearly counter-intuitive because we expect to get $(A \equiv B) > C$.

C. Spearman's footrule method

Spearman's [34] L_1 distance (named also F-distance or footrule in [35]) is the sum of the absolute differences between the rank values of the rankings. Suppose

[14] Preference orderings are called *preferential arrangements* by Gross.

[15] implemented in *nchoosek* function in Matlab™, for instance.

[16] Detailed presentations of voting methods are given in [2] and [27].

[17] actually devised by Llull in 1299, and called also Ranked-Robin method. A voter who leaves some candidates' rankings blank is assumed to be indifferent between them but to prefer all ranked candidates to them.

[18] where any alternative (or candidate) who wins every one-on-one election must have the most victories overall. Borda's method does not always satisfy this Condorcet's principle [29].

we have two ranking vectors $\boldsymbol{r}_1 = [r_1(i), i = 1, \ldots, n]$ and $\boldsymbol{r}_2 = [r_2(i), i = 1, \ldots, n]$ of n alternatives, then Spearman's footrule is defined by

$$F(\boldsymbol{r}_1, \boldsymbol{r}_2) = \sum_{i=1}^{n} |r_1(i) - r_2(i)|. \tag{4}$$

This distance can be normalized by dividing it by $n^2/2$. Rank aggregation based on Spearman's footrule has been developed by Fagin et al. [32], [33] who present heuristic methods to estimate the optimal aggregated ranking in the footrule sense. The main problem with this distance is that it is not invariant under the labeling of the alternatives (or candidates). This means that we get different results depending of how the set of alternatives is defined, see counter-example in [38]. This is we think a serious drawback because the result is not robust to permutations of candidate labels/indexes. That is why we do not recommend and use it.

V. On optimal solution of CRP

In this section we present the optimal theoretical approaches for solving CRP and we will show that they actually fail to provide acceptable (i.e. commonsense, natural or expected) solution in very simple interesting examples. This surprising and disappointing result at the current stage of our investigations clearly reveals the difficulty of solving CRP to get trustable solution for general and more complicated problems involving many TPOs.

The basic idea to get an optimal solution of CRP is simple. Suppose we have a good (i.e. adequate and powerful) metric $d(\text{Pref}_1, \text{Pref}_2)$ able to measure the distance between two preferences orderings Pref_1 and Pref_2, then among the finite set of all possible (total) preference orderings $\mathcal{P}(n)$ the optimal choice in the Least Squares (LS) sense is to select the best preference ordering $\text{Pref}^\star \in \mathcal{P}(n)$ that minimizes the mean distance between the optimal solution $\text{Pref}^\star$ and the set of preference orderings $\{\text{Pref}_1, \ldots, \text{Pref}_S\}$ we have, that is

$$\text{Pref}^\star = \arg \min_{\text{Pref} \in \mathcal{P}(n)} \sqrt{\sum_{s=1}^{S} d^2(\text{Pref}, \text{Pref}_s)}. \tag{5}$$

If one wants to take into account the weights of importance of each sources (i.e. voter) the problem to solve will become to determinate the social choice $\text{Pref}\star \in \mathcal{P}(n)$ such that

$$\text{Pref}^\star = \arg \min_{\text{Pref} \in \mathcal{P}(n)} \sqrt{\sum_{s=1}^{S} u_s d^2(\text{Pref}, \text{Pref}_s)}, \tag{6}$$

where u_s is the given weight of importance of the source of information E_s providing the preference ordering Pref_s.

This problem is a priori difficult to solve because we need to use a good metric $d(\cdot, \cdot)$ to measure the distance between two preference orderings, and because of the high dimension of the discrete search space $\mathcal{P}(n)$ where the optimal solution satisfying (5) (or (6)) must belong.

A. Using classical Kemeny and Frobenius distances

An appealing candidate for $d(\text{Pref}_1, \text{Pref}_2)$ is the well-known Kemeny's distance [37] which is a metric satisfying few very reasonable axioms (including the invariance under labeling desideratum - see condition 2 of [36], p. 587). Kemeny's distance between Pref_1 and Pref_2 is based on the construction of two $n \times n$ pairwise Preference-Score Matrices (PSM) $\boldsymbol{M}_1 = [M_1(i,j)]$ and $\boldsymbol{M}_2 = [M_2(i,j)]$, and the L_1 distance as follows[19]

$$d_K(\text{Pref}_1, \text{Pref}_2) = \frac{1}{2} \sum_{i=1}^{n} \sum_{i=1}^{n} |M_1(i,j) - M_2(i,j)|, \tag{7}$$

where $M_s(i,j)$ for $s = 1, 2$ and $i, j = 1, \ldots, n$ is defined from pairwise comparisons by[20]

$$\boldsymbol{M}_s(i,j) = \begin{cases} 1, & \text{if } x_i \succ x_j \text{ in } \text{Pref}_s, \\ -1, & \text{if } x_i \prec x_j \text{ in } \text{Pref}_s, \\ 0, & \text{if } x_i = x_j \text{ in } \text{Pref}_s. \end{cases} \tag{8}$$

By convention, the row index i of $\boldsymbol{M}_s$ corresponds to the index of elements x_i on the left side of preference order $x_i \succ x_j$ in Pref_s, and the column index j of $\boldsymbol{M}_s$ corresponds to the index of the element x_j on the right side of preference order $x_i \succ x_j$ in Pref_s. Kemeny's distance has been widely used in many applications because of its appealing properties (true metric and invariance under labeling), and mainly because it has been widely claimed to be the unique metric satisfying Kemeny's axioms.

Very recently we have proved in [38] that the unicity of Kemeny's distance is violated and there is another metric that also satisfies Kemeny's axioms which is also invariant under labeling. More specifically, we can use the Frobenius metric defined by the L_2-norm of the matrix

$$||\boldsymbol{M}||_F = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |M(i,j)|^2} = \sqrt{\text{Tr}(\boldsymbol{M}^T \boldsymbol{M})}, \tag{9}$$

where $\boldsymbol{M}^T$ is the transpose of the matrix $\boldsymbol{M}$, and Tr(.) is the trace operator for matrix. Based on this norm, the distance between Pref_1 and Pref_2 induced by the two PSM matrices $\boldsymbol{M}_1$ and $\boldsymbol{M}_2$ is simply defined by[21]

$$d_F(\boldsymbol{M}_1, \boldsymbol{M}_2) = ||\boldsymbol{M}_1 - \boldsymbol{M}_2||_F. \tag{10}$$

The natural question is to know (and verify) if the use of d_K or d_F as defined in (7) and (10) are good candidates to provide optimal solution(s) that fit with what we naturally expect. To answer to this important question, let us examine the results obtained for examples 1, 2 and 3 of the section II.

With the example 1, we have to explore only the 13 possibilities of preference orderings Pref listed in Table II, and

[19] We use the subscript K in our notation to refer to Kemeny.

[20] Instead of using 1, -1, 0 values we can also use 1, 0 and 0.5 values. This will not change the normalized distance $\tilde{d}_K(\text{Pref}_1, \text{Pref}_2) = d_K(\text{Pref}_1, \text{Pref}_2)/d_K^{\max}$, where $d_K^{\max}$ is the maximum distance between two preferences orderings of $\mathcal{P}(n)$.

[21] We use the subscript F in our notation to refer to Frobenius.

we calculate $d_K(\text{Pref}, \text{Pref}_s)$ and $d_F(\text{Pref}, \text{Pref}_s)$ for $s = 1, 2$. This is not a big search space to explore. The KOA solution(s) and for FOA solution(s) are respectively

$$\text{Pref}_K^\star = \arg \min_{\text{Pref} \in \mathcal{P}(n)} \sqrt{\sum_{s=1}^{2} d_K^2(\text{Pref}, \text{Pref}_s)}, \qquad (11)$$

$$\text{Pref}_F^\star = \arg \min_{\text{Pref} \in \mathcal{P}(n)} \sqrt{\sum_{s=1}^{2} d_F^2(\text{Pref}, \text{Pref}_s)}. \qquad (12)$$

Using Kemeny's distance defined in (7) with a simple Matlab™ implementation of this search we get three KOA solutions as follows: $\text{Pref}_K^{\star,1} : C \succ (A \equiv B)$, $\text{Pref}_K^{\star,2} : (A \equiv B) \succ C$, and $\text{Pref}_K^{\star,3} : (A \equiv B \equiv C)$. Using Frobenius distance defined in (10) we get only one FOA solution $\text{Pref}_F^\star : (A \equiv B \equiv C)$. Clearly, we see that the KOA and FOA solutions are not acceptable because with Kemeny's distance we get multiple solutions and we cannot infer a priori which one is correct, and with Frobenius distance the FOA solution does not fit with the expected solution $(A \equiv B) \succ C$.

With the (Condorcet's) example 2, the KOA and FOA solutions are $\text{Pref}_K^\star : (A \equiv B \equiv C)$, and $\text{Pref}_F^\star : (A \equiv B \equiv C)$. These solutions are in agreement with the natural expected social ordering $(A \equiv B \equiv C)$.

With the (majority) example 3, the KOA and FOA solutions are $\text{Pref}_K^\star : A \succ (B \equiv C)$, and $\text{Pref}_F^{\star,1} : A \succ B \succ C$ and $\text{Pref}_F^{\star,2} : A \succ (B \equiv C)$. These solutions are in disagreement with the natural expected social ordering $A \succ B \succ C$.

In summary, these simple counterexamples (i.e. example 1 and example 3) suffice to show that the optimal KOA (resp. FOA) solutions based on basic Kemeny's (resp. Frobenius) distance using PSM defined in (8) go against expected solutions in some situations. In fact there is no guarantee that ranking aggregation based on these distances and LS approach will give an acceptable solution in general even if it is optimal in the LS sense. Therefore, a better optimal approach must be sought to obtain more acceptable aggregated ranking solution.

B. Testing the weighted Kemeny and Frobenius distances

After analyzing the unexpected optimal solutions provided by the classical Kemeny and Frobenius distances we suspected that the problem comes from the definition of PSM which does not take properly into account the rank of alternatives when making pairwise comparison as done in (8). We thought that this missing important information was the origin of the problem that generates unexpected KOA and FOA solutions. To try to circumvent this drawback, we first attempted to use the importance weighting vector reflecting the intensity of the rank of each alternative in the preference ordering as proposed in [39]. For this, we have tested the weighting vector $\boldsymbol{w}^{(2)} \triangleq [1/2, 1/4, 1/8, \ldots, 1/2^n]$ which means that the alternative at rank 1 has a weight equal to 1/2, the alternative at rank 2 has a weight equal to 1/4, and so on. Then we apply the weighted Kemeny's distance $d_{K,\boldsymbol{w}}$ and the weighted Frobenius distance $d_{F,\boldsymbol{w}}$ defined in [38] for the search of optimal KOA and FOA solutions. This attempt unfortunately does not work for examples 1 and 2. The same conclusion is drawn when using the $\boldsymbol{w}^{(1)}$ vector defined by $\boldsymbol{w}^{(1)} \triangleq [1, 1/2, 1/3, \ldots, 1/n]$. Therefore the optimal solutions obtained by the weighted Kemeny and weighted Frobenius distances do not always agree with the natural expected solutions. So, we have explored a bit further the question as follows.

C. Proposal for a new definition of PSM

Actually the weighting method proposed in [39] uses the matrix product $\boldsymbol{W} \cdot \boldsymbol{M}$ in the weighted Kemeny and weighted Frobenius distances, and this product of matrices does not generate an anti-symmetrical matrix in general (the matrix $\boldsymbol{W}$ being equal to $diag(\boldsymbol{w})$).

For instance, if the (ordered) set of objects is $\{A, B, C\}$ and if $\text{Pref}_s : B \succ C \succ A$ then $\boldsymbol{r}_s = [3, 1, 2]$ and by taking $\boldsymbol{w}_s \triangleq [1/3, 1, 1/2]$ we will obtain a non-anti-symmetrical matrix

$$\boldsymbol{W} \cdot \boldsymbol{M} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} \cdot \begin{bmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{3} & -\frac{1}{3} \\ 1 & 0 & 1 \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{bmatrix}$$

In order to deal with weighted PSM satisfying the anti-symmetrical property we propose to change the definition of PSMs given in (8) for taking into account the rank of alternatives. More precisely, we propose the following definition

$$\boldsymbol{M}_s(i, j) \triangleq w_i(\boldsymbol{r}_s) - w_j(\boldsymbol{r}_s) \qquad (13)$$

where $\boldsymbol{r}_s$ is the ranking vector associated to the preference ordering Pref_s, and $w_i(\boldsymbol{r}_s) = 1/r_s(i)$ and $w_j(\boldsymbol{r}_s) = 1/r_s(j)$ are the weights of the i-th and j-th alternatives.

Considering again the (ordered) set of objects is $\{A, B, C\}$ with $\text{Pref}_s : B \succ C \succ A$. We have $\boldsymbol{r}_s = [3, 1, 2]$. Hence, $w_1 = 1/3$, $w_2 = 1/1$ and $w_3 = 1/2$ and the new PSM matrix based on definition (13) will be

$$\boldsymbol{M}_s = \begin{bmatrix} 0 & \frac{1}{3} - 1 & \frac{1}{3} - \frac{1}{2} \\ 1 - \frac{1}{3} & 0 & 1 - \frac{1}{2} \\ \frac{1}{2} - \frac{1}{3} & \frac{1}{2} - 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{2}{3} & -\frac{1}{6} \\ \frac{2}{3} & 0 & \frac{1}{2} \\ \frac{1}{6} & -\frac{1}{2} & 0 \end{bmatrix}$$

which is as we can easily verify an anti-symmetrical matrix.

If we prefer using $\boldsymbol{w}^{(2)} \triangleq [1/2, 1/4, 1/8, \ldots, 1/2^n]$ weighting vector [39], then we will have $w_1 = 1/8$, $w_2 = 1/2$ and $w_3 = 1/4$ and the new PSM matrix based on definition (13) will be

$$\boldsymbol{M}_s = \begin{bmatrix} 0 & \frac{1}{8} - \frac{1}{2} & \frac{1}{8} - \frac{1}{4} \\ \frac{1}{2} - \frac{1}{8} & 0 & \frac{1}{2} - \frac{1}{4} \\ \frac{1}{4} - \frac{1}{8} & \frac{1}{4} - \frac{1}{2} & 0 \end{bmatrix} = \begin{bmatrix} 0 & -\frac{3}{8} & -\frac{1}{8} \\ \frac{3}{8} & 0 & \frac{1}{4} \\ \frac{1}{8} & -\frac{1}{4} & 0 \end{bmatrix}$$

which is also an anti-symmetrical matrix.

Based on the new weighted PSM definition (13) and the direct use of $\boldsymbol{w}_s = 1/\boldsymbol{r}_s$ (even when ties occur in the preference ordering) gives correct/expected solution for examples 1 and 2 with the FOA. The KOA does not give the correct/expected solution for example 1 but it gives the correct solution for example 2. Unfortunately the KOA and FOA do not give the correct solution for example 3. Because of these disappointing results we have also explored two methods for determining weights when ties occur in a preference ordering Pref_s as follows:

- Balanced-method: in this method the ranks of tied alternatives in $\boldsymbol{r}_s$ are just averaged to get a balanced ranking vector $\boldsymbol{r}'_s$ within interval $[1,n]$ (n being the number of alternatives, and then we take $\boldsymbol{w}'_s = 1/\boldsymbol{r}'_s$ as vector of weights. For instance, consider six ($n=6$) alternatives A, B, C, D, E and F with $\text{Pref}_s : (A \equiv B \equiv C) \succ F \succ (D \equiv E)$. Then this tied ranking vector $\boldsymbol{r}_s = [1,1,1,3,3,2]$ is balanced/adjusted with respect to the scale $[1,6]$ as follows

  $$\boldsymbol{r}'_s = [\frac{1+2+3}{3}, \frac{1+2+3}{3}, \frac{1+2+3}{3}, \frac{5+6}{2}, \frac{5+6}{2}, 4] \\ = [2,2,2,5.5,5.5,4]$$

  Hence the modified weighting vector will be

  $$\boldsymbol{w}'_s = 1/\boldsymbol{r}'_s \approx [0.5, 0.5, 0.5, 0.18, 0.18, 0.25].$$

- Readjusting-method: in this method we divide the rank value of tied alternative by their multiplicity and we readjust their rank. For instance, consider six ($n=6$) alternatives A, B, C, D, E and F with $\text{Pref}_s : (A \equiv B \equiv C) \succ F \succ (D \equiv E)$. The weights of tied alternatives A, B and C at rank 1 are set to 1/3, the following ranked 4th alternatives at rank 2 is F so we set its weight to 1/4, and the following ranked tied alternatives at rank 3 are D and E so we set their weights to 1/5. With this method and for this example we will use the modified weighting vector

  $$\boldsymbol{w}'_s = [\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{5}, \frac{1}{5}, \frac{1}{4}].$$

D. Optimal solution based on the new PSM definition

Based on the balanced-method of setting the weights and the new PSM definition (13), we get:

- For example 1 (Zadeh's alike), we obtain

  $$\text{Pref}^{\star,1}_K : (A \equiv B) \succ C), \text{and } \text{Pref}^{\star,2}_K : (A \equiv B \equiv C), \\ \text{Pref}^{\star}_F : (A \equiv B \equiv C),$$

  whereas the expected solution is $(A \equiv B) \succ C)$. In this example, KOA solution is multiple and FOA solution does not fit with the natural expected solution.
- For example 2 (Condorcet), we obtain

  $$\text{Pref}^{\star}_K : (A \equiv B \equiv C), \\ \text{Pref}^{\star}_F : (A \equiv B \equiv C).$$

  The expected solution is $(A \equiv B \equiv C)$. In this example KOA and FOA provide the correct expected solution.
- For example 3 (Majority), we obtain

  $$\text{Pref}^{\star}_K : A \succ (B \equiv C), \\ \text{Pref}^{\star}_F : A \succ B \succ C.$$

  The expected solution is $A \succ B \succ C$, and we see that only FOA gives the correct expected solution.
- For example 4, we obtain

  $$\text{Pref}^{\star}_K : (A \equiv B \equiv C \equiv D), \\ \text{Pref}^{\star}_F : (A \equiv B \equiv D) \succ C)$$

  whereas the expected solution is $(A \equiv B \equiv D) \succ C$. In this example only FOA provides a solution in agreement with the expected solution.

Based on the balanced-method for the weights, we see that KOA and FOA are not able to provide a correct solution for all the four simple examples analyzed in this paper.

Based on the readjusting-method of setting the weights and the new PSM defintion (13), we get:

- For example 1 (Zadeh's alike), we obtain

  $$\text{Pref}^{\star,1}_K : (A \equiv B) \succ C, \text{and } \text{Pref}^{\star,2}_K : (A \equiv B \equiv C), \\ \text{Pref}^{\star}_F : (A \equiv B) \succ C,$$

  whereas the expected solution is $(A \equiv B) \succ C)$. In this example KOA solution is multiple, and FOA solution fits with the natural expected solution.
- For example 2 (Condorcet), we obtain

  $$\text{Pref}^{\star}_K : (A \equiv B \equiv C), \\ \text{Pref}^{\star}_F : (A \equiv B \equiv C).$$

  The expected solution is $(A \equiv B \equiv C)$. In this example KOA and FOA provide the correct expected solution.
- For example 3 (Majority), we obtain

  $$\text{Pref}^{\star}_K : A \succ C \succ B, \\ \text{Pref}^{\star}_F : (A \equiv C) \succ B.$$

  The expected solution is $A \succ B \succ C$, and we see that KOA and FOA do not provide solutions in agreement with the expected solution.
- For example 4

  $$\text{Pref}^{\star}_K : (A \equiv B) \succ C \succ D, \\ \text{Pref}^{\star}_F : (A \equiv B) \succ D \succ C,$$

  whereas the unique expected solution is $(A \equiv B \equiv D) \succ C$. In this example KOA and FOA do not provide solutions in agreement with the expected solution.

Based on the readjusting-method for the weights, we see that KOA and FOA are also not able to provide a correct solution for all the four simple examples analyzed in this paper.

The results of these two analyses are disappointing because we see that even for these quite simple examples the KOA and FOA fail to provide expected results. Our work clearly shows that the solution of CRP is much more difficult to solve than anticipated based on KOA and FOA using least squares criterion.

VI. Conclusions, perspectives and challenges

In this work we have shown why classical methods (Borda's count, Copeland's method and Spearman's footrule) are not satisfactory to make the ranking aggregation. Based on this matter of fact we have analyzed how a search method based on metrics between preference orderings and minimization of least squares criterion should be in theory possible. A deep analysis in this direction using Kemeny's and Frobenius distances reveals that the optimal solutions we get do not always fit with the expected solution for different interesting examples. While this result is disappointing at the current stage of this research work it is no vain because it casts in doubts the pertinence or validity of some results of the literature based on the Kemeny's optimal approach. This work clearly points out that the CRP optimal solution must be explored in deep and cannot be used without the guarantee that the "optimal" solution makes perfectly sense. This is an open challenging question. Moreover, even if an optimal powerful approach is found for solving the CRP the second important challenge will be to apply it with big discrete search spaces whose cardinalities follow ordered Bell numbers. We will have also to be able to extend the method to the case of partial preference orderings (PPO) based on a suitable distance between PPOs which will be presented in a forthcoming publication. This will involve to deal with much greater combinatorics. The optimization problem addressed in this paper is of prime importance because if it can be solved efficiently it could offer a better foundation for the voting system in democracies, as well as better techniques for decision-making based on multiple criteria as well because the ranking formalism is the simplest ways to provide information for making a decision. In this case each criteria can be considered as a *voter* providing a preference order among the different alternatives (interpreted as *candidates* running in an election).

References

[1] S.J. Brams, *Mathematics and Democracy: Designing Better Voting and Fair-Division Procedures*, Princeton University Press, 2008.

[2] P. Tannenbaum, *Excursions in Modern Mathematics*, (8th Edition), Pearson New International Edition, 2014.

[3] J.C. de Borda, *Mémoire sur les Élections au Scrutin*, Histoire de l'Académie Royale des Sciences, Paris. Translated by Alfred de Grazia in 1953, (Mathematical Derivation of an Election System), Isis, 44 (1–2), pp. 42–51, 1781.

[4] Marquis de Condorcet, *Essai sur l'application de l'analyse à la probabilité des décisions rendues à la pluralité des voix*, Paris, France, 1785.

[5] K.J. Arrow, *Social Choice and Individual Values*, (3rd Edition), Yale University Press, 2012. (see also https://plato.stanford.edu/entries/arrows-theorem/)

[6] A. Sen, *Collective choice and social welfare*, San Francisco: Holden-Day, 1970.

[7] H. Bury, D. Wagner, *Group judgement with ties. A position-based approach*, Operations Research and Decisions, Vol. 19(4), pp. 7–26, 2009.

[8] L.A. Zadeh, *On the validity of Dempster's rule of combination*, Memo M79/24, Univ. of California, Berkeley, USA, 1979.

[9] G. Shafer, *A mathematical theory of evidence*, Princeton Univ. Press, 1976.

[10] J. Dezert, P. Wang, A. Tchamova, *On The Validity of Dempster-Shafer Theory*, in Proc. of Fusion 2012, Singapore, July 2012.

[11] A. Tchamova, J. Dezert, *On the Behavior of Dempster's Rule of Combination and the Foundations of Dempster-Shafer Theory*, Proc. of IEEE IS'2012, Sofia, Bulgaria, Sept. 6–8, 2012.

[12] Z. Kereskényi-Balogh, G. Nyul, *Fubini Numbers and Polynomials of Graphs*, Mediterranean. J. of Math., Vol. 18, Article no 230, 2021.

[13] A. Cayley, *On the analytical forms called trees - Second part*, Philosophical Magazine, Vol. XVIII, pp. 374–378, 1859.

[14] L. Comtet, *Advanced Combinatorics: The Art of Finite and Infinite Expansions*, p. 228 #20, D. Reidel Publishing Co., 1974.

[15] N.J.A. Sloane, *A Handbook of Integer Sequences*, Academic Press, 1973.

[16] N.J.A. Sloane (Ed.), *The On-line Encyclopedia of Integer Sequences*, 2024. https://oeis.org/A000670

[17] N.J.A. Sloane, *Handbook of Integer Sequences - Fifty Years Later*, 2023. https://arxiv.org/pdf/2301.03149.pdf

[18] O.A. Gross, *Preferential Arrangements*, The American Mathematical Monthly, Vol. 69(1), pp. 4–8, 1962.

[19] N. Pippenger, *The Hypercube of Resistors, Asymptotic Expansions, and Preferential Arrangements*, Mathematics Magazine, Vol. 83(5), pp. 331–346, Dec. 2010.

[20] E. Mendelson, *Races with Ties*, Mathematics Magazine, Vol. 55(3), pp. 170–175, May, 1982.

[21] I.J. Good, *The number of orderings of n candidates when ties are permitted*, Fibonacci Quarterly, 13: 11–18, 1975.

[22] M. Mor, A.S. Fraenkel, *Cayley permutations*, Discrete Mathematics, Vol. 48(1), pp. 101–112, 1984.

[23] B. Luong *Set partition*, MATLAB Central File Exchange, Retrieved February 7, 2024. https://www.mathworks.com/matlabcentral/fileexchange/24133-set-partition

[24] Sage Mathematical Software System (available on GitHub platform). https://github.com/sagemath

[25] C. Dwork, R. Kumar, M. Naor, D. Sivakumar, *Rank aggregation methods for the web*, in Proc. of the 10th Int. Conf. on World Wide Web, pp.613–622, 2001,

[26] A.H. Copeland, *A 'reasonable' social welfare function*, Seminar on Mathematics in Social Sciences, University of Michigan (unpublished), 1951. https://en.wikipedia.org/wiki/Copeland's_method

[27] https://www.equal.vote/

[28] H.P. Young, *Condorcet's theory of voting*, American Political Science Review, Vol. 82, pp. 1231–1244, 1988.

[29] H.P. Young, A. Levenglick, *A consistent extension of Condorcet's election principle*, ORSAITIMS National Meeting at Miami, FL, USA, Nov. 4th, 1976 (also in SIAM Journal on Applied Mathematics, Vol. 35(2), pp. 285–300, 1978).

[30] A. Argentini, *Ranking aggregation based on belief function theory*, PhD Thesis, DISI - University of Trento, Italy, March 2012.

[31] R. Fagin, A. Lotem, M. Naor, *Optimal aggregation algorithms for middleware*, Proc. of 20th ACM Symp. on Principles of Database Systems, pp. 102–113, 2001.

[32] R. Fagin, R. Kumar, D. Sivakumar, *Comparing top k Lists*, Proc. of 14th Annual ACM-SIAM Symp. on Discrete Algorithms, pp. 28–36, 2003.

[33] R. Fagin, R. Kumar, D. Sivakumar, *Efficient Similarity Search and Classification via Rank Aggregation*, Proc. of SIGMOD 2003, San Diego, CA, USA, June 9–12, 2003.

[34] C. Spearman, *The proof and measurement of association between two things*, American J. of Psychology (AJP), Vol. 100, pp. 441–471, 1961.

[35] P. Diaconis, R.L. Graham, *Spearman's footrule as a measure of disarray*, J. of the Royal Stat. Soc. Series B (Methodological), Vol. 30(2), pp. 262–268, 1979.

[36] J.G. Kemeny, *Mathematics without numbers*, Daedalus, Vol. 88, pp.577–591, 1959.

[37] J.G. Kemeny, J.L. Snell, *Mathematical Models in the Social Sciences*, (1st Edition Ginn, Boston, 1962), Chap. 2, MIT Press Edition, 141 pages, 1972.

[38] J. Dezert, A. Shekhovtsov, W. Sałabun, *A new distance between rankings*, Heliyon 10, e28265, March 2024. https://doi.org/10.1016/j.heliyon.2024.e28265

[39] W. Sałabun, K. Urbaniak, *A new coefficient of rankings similarity in decision-making problems*, Proc. of Int. Conf. on Computational Science (ICCS 2020), Part II 20, pp. 632–645, Amsterdam, The Netherlands, June 3–5, 2020.